3 The integral

3.1 The integral of simple functions

Definition 3.1. Let X be a measure space with measure μ. A simple function $X \to \mathbb{K}$ is called *integrable* iff it vanishes outside of a set of finite measure. We denote the vector space of integrable simple functions on X with respect to the measure μ by $\mathcal{S}(X, \mu)$.

Exercise 20. Show that the integrable simple functions actually form an algebra over $\mathbb{K}$.

Definition 3.2. Let S be a measure space with measure μ. A (μ-)*integral* is a collection of linear maps

$$\mathcal{S}(X, \mu) \to \mathbb{K} : f \mapsto \int_X f \, \mathrm{d}\mu,$$

one for each measurable subset $X \subseteq S$, satisfying the following properties:

- If X has finite measure, then $\int_X 1 \, \mathrm{d}\mu = \mu(X)$, where $1 \in \mathcal{S}(X, \mu)$ is the constant function with value 1.
- If $X_1, X_2 \subseteq X$ are measurable such that $X_1 \cap X_2 = \emptyset$ and $X_1 \cup X_2 = X$, and $f \in \mathcal{S}(X, \mu)$ then $\int_X f \, \mathrm{d}\mu = \int_{X_1} f \, \mathrm{d}\mu + \int_{X_2} f \, \mathrm{d}\mu$.

Proposition 3.3. *The integral exists and is unique.*

Proof. **Exercise.** □

When it is clear with respect to which measure the integral is taken, the symbol $\mathrm{d}\mu$ may be omitted. When the integral is taken with respect to the whole measure space and it is clear which measure space this is, the subscript indicating the set over which is integrated may be omitted.

Proposition 3.4. *The integral of integrable simple maps has the following properties:*

- *If f and g are real valued and $f(x) \leq g(x)$ for all $x \in X$, then $\int_X f \leq \int_X g$.*
- *If $f(x) \geq 0$ for all $x \in X$ and $A \subseteq X$ is measurable, then $\int_A f \leq \int_X f$.*
- $\left|\int_X f\right| \leq \int_X |f|$.
- *Suppose X has finite measure, then $\int_X |f| \leq \|f\|_{sup} \, \mu(X)$. (Here $\|\cdot\|_{sup}$ denotes the supremum norm.)*

Proof. **Exercise.** □

Proposition 3.5. *The space $\mathcal{S}(X,\mu)$ carries a seminorm given by*

$$\|f\|_1 := \int_X |f| \, \mathrm{d}\mu.$$

Proof. **Exercise.** □

The fact that we only have a seminorm and not necessarily a norm comes from the inability of the integral to "see" sets of measure zero.

Proposition 3.6. *Let $f \in \mathcal{S}(X,\mu)$. Then, $\|f\|_1 = 0$ iff f vanishes outside a set of measure zero.*

Proof. **Exercise.** □

We also say "almost everywhere" to mean "outside a set of measure zero".

Lemma 3.7. *Let $(X,\mathcal{M},\mu)$ be a measure space and $\mathcal{N}$ an algebra of subsets of X that generates the σ-algebra $\mathcal{M}$. Let $f \in \mathcal{S}(X,\mu)$ and $\epsilon > 0$. Then, there exists $g \in \mathcal{S}(X,\mu)$ such that g is measurable with respect to $\mathcal{N}$ (i.e., $g^{-1}(\{p\}) \subseteq \mathcal{N}$ for all $p \in \mathbb{K}$) and such that $\|f-g\|_1 < \epsilon$.*

Proof. **Exercise.**Hint: Use Proposition 2.37. □

3.2 Integrable functions

Lemma 3.8. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence of elements of $\mathcal{S}(X,\mu)$ with respect to the seminorm $\|\cdot\|_1$. Then, there exists a subsequence which converges pointwise almost everywhere to some measurable map f and for any $\epsilon > 0$ converges uniformly to f outside of a set of measure less than ϵ.*

Proof. Since $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy, there exists a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ such that

$$\|f_{n_l} - f_{n_k}\|_1 < 2^{-2k} \quad \forall k \in \mathbb{N} \quad \text{and} \quad \forall l \geq k.$$

Define

$$Y_k := \{x \in X : |f_{n_{k+1}}(x) - f_{n_k}(x)| \geq 2^{-k}\} \quad \forall k \in \mathbb{N}.$$

Then,

$$2^{-k}\mu(Y_k) \leq \int_{Y_k} |f_{n_{k+1}} - f_{n_k}| \leq \int_X |f_{n_{k+1}} - f_{n_k}| \leq 2^{-2k} \quad \forall k \in \mathbb{N}.$$

This implies, $\mu(Y_k) \leq 2^{-k}$ for all $k \in \mathbb{N}$. Define now $Z_j := \bigcup_{k=j}^{\infty} Y_k$ for all $j \in \mathbb{N}$. Then, $\mu(Z_j) \leq 2^{1-j}$ for all $j \in \mathbb{N}$.

Fix $\epsilon > 0$ and choose $j \in \mathbb{N}$ such that $2^{1-j} < \epsilon$. Let $x \in X \setminus Z_j$. Then, for $k \geq j$ we have

$$|f_{n_{k+1}}(x) - f_{n_k}(x)| < 2^{-k}.$$

Thus, the sum $\sum_{k=1}^{\infty} f_{n_{k+1}}(x) - f_{n_k}(x)$ converges absolutely. In particular, the limit

$$f(x) := \lim_{l\to\infty} f_{n_l}(x) = f_{n_1}(x) + \sum_{l=1}^{\infty} f_{n_{l+1}}(x) - f_{n_l}(x)$$

exists. For all $k \geq j$ we have the estimate,

$$|f(x) - f_{n_k}(x)| = \left|\sum_{l=k}^{\infty} f_{n_{l+1}}(x) - f_{n_l}(x)\right| \leq \sum_{l=k}^{\infty} \left|f_{n_{l+1}}(x) - f_{n_l}(x)\right| \leq 2^{1-k}$$

Thus, $\{f_{n_k}\}_{k\in\mathbb{N}}$ converges to f uniformly outside of Z_j, where $\mu(Z_j) < \epsilon$.

Repeating the argument for arbitrarily small ϵ we find that f is defined on $X \setminus Z$, where $Z := \bigcap_{j=1}^{\infty} Z_j$. Furthermore, $\{f_{n_k}\}_{k\in\mathbb{N}}$ converges to f pointwise on $X \setminus Z$. Note that $\mu(Z) = 0$. By Theorem 2.19, f is measurable on $X \setminus Z$. We extend f to a measurable function on all of X by declaring $f(x) = 0$ if $x \in Z$. This completes the proof. □

Lemma 3.9. *Let $\{f_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ be Cauchy sequences of elements of $\mathcal{S}(X,\mu)$ with respect to the seminorm $\|\cdot\|_1$. Furthermore, assume that both sequences converge pointwise almost everywhere to the same measurable function f. Then, the following limits exist and are equal,*

$$\lim_{n\to\infty} \int_X f_n = \lim_{n\to\infty} \int_X g_n.$$

Proof. It is easy to see that both limits exist (**<u>Exercise</u>.**). It remains to show that they are equal. To this end consider the sequence formed by the differences $h_n := f_n - g_n$. Then, $\{h_n\}_{n\in\mathbb{N}}$ is a $\|\cdot\|_1$-Cauchy sequence that converges pointwise almost everywhere to zero. We need to show that the limit $\lim_{n\to\infty} \int_X h_n$ (which we already know to exist) is equal to zero.

By Lemma 3.8 there exists a subsequence $\{h_{n_k}\}_{k\in\mathbb{N}}$ with the following property: For any $\delta > 0$ there exists a set Z_δ with $\mu(Z_\delta) < \delta$ such that the subsequence converges absolutely and uniformly to 0 on $X \setminus Z_\delta$.

Choose $\epsilon > 0$ arbitrary. There exists $m \in \mathbb{N}$ such that $\|h_n - h_m\|_1 < \epsilon$ for all $n \geq m$. Let A be a set of finite measure, so that h_m vanishes outside of A. Then,

$$\int_{X\setminus A} |h_n| = \int_{X\setminus A} |h_n - h_m| \leq \int_X |h_n - h_m| < \epsilon \quad \forall n \geq m.$$

Set $\delta := \epsilon/(1 + \|h_m\|_{\sup})$ and $\xi := \epsilon/(1 + \mu(A))$. Then, there exists $l \in \mathbb{N}$ such that $n_l \geq m$ and $|h_{n_k}(x)| < \xi$ for all $k \geq l$ and $x \in X \setminus Z_\delta$. This implies,

$$\int_{A\setminus Z_\delta} |h_{n_k}| \leq \mu(A \setminus Z_\delta)\,\xi \leq \mu(A)\,\xi < \epsilon \quad \forall k \geq l.$$

On the other hand,

$$\int_{Z_\delta} |h_n| \le \int_{Z_\delta} |h_n - h_m| + \int_{Z_\delta} |h_m| \\ \le \|h_n - h_m\|_1 + \mu(Z_\delta)\,\|h_m\|_{\sup} < 2\epsilon \quad \forall n \ge m.$$

Taking the three integral estimates together we get

$$\left|\int_X h_{n_k}\right| \le \int_X |h_{n_k}| \le \int_{X\setminus A} |h_{n_k}| + \int_{A\setminus Z_\delta} |h_{n_k}| + \int_{Z_\delta} |h_{n_k}| < 4\epsilon \quad \forall k \ge l.$$

Since ϵ was arbitrary, we conclude

$$\lim_{n\to\infty} \int_X h_n = \lim_{k\to\infty} \int_X h_{n_k} = 0.$$

□

We are now ready to define the integral more generally.

Definition 3.10. A measurable map f on X is called *integrable* iff there exists a $\|\cdot\|_1$-Cauchy sequence of integrable simple maps that converges pointwise to f almost everywhere. We denote the vector space of integrable maps with values in $\mathbb{K}$ by $\mathcal{L}^1(X,\mu,\mathbb{K})$.

Exercise 21. Show that the integrable functions actually form a vector space.

Definition 3.11. Let $f \in \mathcal{L}^1(X,\mu)$ and $\{f_n\}_{n\in\mathbb{N}}$ a Cauchy sequence of elements of $\mathcal{S}(X,\mu)$ that converges pointwise to f almost everywhere. We define the (μ-)*integral* of f on X by

$$\int_X f := \lim_{n\to\infty} \int_X f_n.$$

That this definition is well follows immediately from Lemma 3.9.

Proposition 3.12. *Let f, g be measurable maps and $f = g$ almost everywhere. Then f is integrable iff g is integrable. Moreover, then,*

$$\int f = \int g.$$

Proof. **Exercise.** □

Proposition 3.13. *Let f be an integrable map. Then, f vanishes outside a σ-finite set.*

Proof. **Exercise.** □

Lemma 3.14. *Let $f \in \mathcal{L}^1(X,\mu)$ and $\{f_n\}_{n\in\mathbb{N}}$ a Cauchy sequence in $\mathcal{S}(X,\mu)$ which converges pointwise to f almost everywhere. Then, $|f| \in \mathcal{L}^1(X,\mu)$ and $\{|f_n|\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{S}(X,\mu)$ which converges pointwise to $|f|$ almost everywhere.*

Proof. **Exercise.** □

Proposition 3.15. *The space $\mathcal{L}^1(X,\mu)$ carries a seminorm given by*

$$\|f\|_1 := \int_X |f| \, \mathrm{d}\mu.$$

Proof. **Exercise.** □

Proposition 3.16. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence of elements of $\mathcal{S}(X,\mu)$ converging pointwise to $f \in \mathcal{L}^1(X,\mu)$ almost everywhere. Then, $\{f_n\}_{n\in\mathbb{N}}$ converges to f in the $\|\cdot\|_1$-seminorm. In particular, every integrable map can be approximated arbitrarily well with respect to the $\|\cdot\|_1$-seminorm by integrable simple maps.*

Proof. Fix $\epsilon > 0$. Since $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy there exists $k \in \mathbb{N}$ such that $\|f_n - f_m\|_1 < \epsilon$ for all $n, m \geq k$. Fix now some $n \geq k$. Then, $\{|f_n - f_m|\}_{m\in\mathbb{N}}$ is a Cauchy sequence of integrable simple maps and converges pointwise almost everywhere to the integrable map $|f_n - f|$. (Use Lemma 3.14.) So, using the definition of the integral,

$$\|f_n - f\|_1 = \int_X |f_n - f| = \lim_{m\to\infty} \int_X |f_n - f_m| = \lim_{m\to\infty} \|f_n - f_m\|_1 \leq \epsilon.$$

This implies the statement. □

Theorem 3.17. *The space $\mathcal{L}^1(X,\mu)$ is complete with respect to the seminorm $\|\cdot\|_1$.*

Proof. Consider a Cauchy sequence $\{f_n\}_{n\in\mathbb{N}}$ in $\mathcal{L}^1(X,\mu)$. Using Proposition 3.16 there is a sequence $\{g_n\}_{n\in\mathbb{N}}$ in $\mathcal{S}(X,\mu)$ such that $\|f_n - g_n\| < 1/n$ for all $n \in \mathbb{N}$. It is easy to see that $\{g_n\}_{n\in\mathbb{N}}$ is Cauchy. (**Exercise.**Show this!) By Lemma 3.8 there is a subsequence $\{g_{n_k}\}_{k\in\mathbb{N}}$ which converges pointwise almost everywhere to an integrable function f. Again using Proposition 3.16 this implies that $\{g_{n_k}\}_{k\in\mathbb{N}}$ converges to f in the $\|\cdot\|_1$-seminorm. But since $\{g_n\}_{n\in\mathbb{N}}$ is Cauchy, by Proposition 1.42 it must also converge to f in the $\|\cdot\|_1$-seminorm. In particular, for $\epsilon > 0$ there exists $k \in \mathbb{N}$ such that $\|f - g_n\|_1 < \epsilon/2$ for all $n \geq k$. But then, for all $n \geq \sup\{k, 2/\epsilon\}$ we have

$$\|f - f_n\|_1 \leq \|f - g_n\|_1 + \|g_n - f_n\|_1 < \epsilon/2 + 1/n \leq \epsilon.$$

That is, $\{f_n\}_{n\in\mathbb{N}}$ converges to f in the $\|\cdot\|_1$-seminorm. □

3.3 Elementary properties of the integral

Proposition 3.18. *The integral of integrable maps has the following properties:*

- *If X_1, X_2 are measurable such that $X = X_1 \cup X_2$ and $X_1 \cap X_2 = \emptyset$ then $\int_X f = \int_{X_1} f + \int_{X_2} f$*
- *If f and g are real valued and $f(x) \le g(x)$ for almost all $x \in X$, then $\int_X f \le \int_X g$.*
- *If f and g are real valued and integrable, then $\sup(f, g)$ and $\inf(f, g)$ are integrable.*
- $\left|\int_X f\right| \le \int_X |f|$.
- *Suppose X has finite measure and f is bounded, then $\int_X |f| \le \|f\|_{sup}\, \mu(X)$.*

Proof. **Exercise.** □

Proposition 3.19. *Let X be a measurable space, $f : X \to \mathbb{R}$, $g : X \to \mathbb{R}$ maps. Then, $f + \mathrm{i}g : X \to \mathbb{C}$ is integrable iff f and g are integrable.*

Proof. **Exercise.** □

Theorem 3.20 (Averaging Theorem)**.** *Let X be a measure space with σ-finite measure μ. Let $S \subseteq \mathbb{K}$ be a closed subset and $f \in \mathcal{L}^1(X, \mu, \mathbb{K})$. If for any measurable set A of finite and positive measure we have*

$$\frac{1}{\mu(A)} \int_A f \mathrm{d}\mu \in S,$$

then $f(x) \in S$ for almost all $x \in X$.

Proof. Let $C := \{x \in X : f(x) \notin S\}$. We need to show that $\mu(C) = 0$. Assume the contrary, i.e., $\mu(C) > 0$. Write $\mathbb{K} \setminus S = \bigcup_{n \in \mathbb{N}} B_n$ as a countable union of closed balls $\{B_n\}_{n \in \mathbb{N}}$. (Use second countability of $\mathbb{K}$ and recall Proposition 1.36.) Their preimages are measurable and cover C. There is at least one closed ball B_n such that $\mu(f^{-1}(B_n)) > 0$. Say this closed ball has center x and radius r. Furthermore, there is a measurable subset $D \subseteq f^{-1}(B_n)$ such that $0 < \mu(D) < \infty$. Then,

$$\begin{aligned}\left|\frac{1}{\mu(D)} \int_D f \,\mathrm{d}\mu - x\right| &= \frac{1}{\mu(D)} \left|\int_D (f - x) \,\mathrm{d}\mu\right| \\ &\le \frac{1}{\mu(D)} \int_D |f - x| \,\mathrm{d}\mu \le \frac{1}{\mu(D)} \int_D r \,\mathrm{d}\mu = r.\end{aligned}$$

In particular, $\frac{1}{\mu(D)} \int_D f \,\mathrm{d}\mu \in B_n$. But $B_n \cap S = \emptyset$, so we get a contradiction with the assumptions. □

Exercise 22. 1. Explain where in the above proof σ-finiteness was used. 2. Extend the proof to the case where μ is not σ-finite by replacing $f(x) \in S$ with $f(x) \in S \cup \{0\}$ in the statement of the Theorem.

Proposition 3.21. *Let $f \in \mathcal{L}^1$ and assume $\int_A f = 0$ for all measurable sets A. Then, $f = 0$ almost everywhere.*

Proof. **Exercise.** □

Proposition 3.22. *Let f be an integrable function. Then, $\|f\|_1 = 0$ iff $f = 0$ almost everywhere.*

Proof. **Exercise.** □

Proposition 3.23. *Let $(X, \mathcal{M}, \mu)$ be a measure space and $\mathcal{N}$ an algebra of subsets of X that generates the σ-algebra $\mathcal{M}$. Let $\mathcal{M}^*$ denote the completion of $\mathcal{M}$ with respect to μ. Let $f \in \mathcal{L}^1(X, \mathcal{M}^*, \mu)$ and $\epsilon > 0$. Then, there exists $g \in \mathcal{S}(X, \mu)$ such that g is measurable with respect to $\mathcal{N}$ and such that $\|f - g\|_1 < \epsilon$.*

Proof. This is clear from combining Proposition 3.16 with Lemma 3.7. □

3.4 Integrals and limits

Theorem 3.24. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{L}^1(X, \mu)$ converging to $f \in \mathcal{L}^1(X, \mu)$ in the $\|\cdot\|_1$-seminorm. Then, there exists a subsequence which converges pointwise almost everywhere to f and for any $\epsilon > 0$ converges uniformly to f outside of a set of measure less than ϵ.*

Proof. We first consider the special case $f = 0$. The proof proceeds in a way similar to that of Lemma 3.8. Consider a subsequence such that

$$\|f_{n_k}\|_1 < 2^{-2k} \quad \forall k \in \mathbb{N}.$$

Define

$$Y_k := \{x \in X : |f_{n_k}(x)| \geq 2^{-k}\} \quad \forall k \in \mathbb{N}.$$

Then,

$$2^{-k}\mu(Y_k) \leq \int_{Y_k} |f_{n_k}| \leq \int_X |f_{n_k}| \leq 2^{-2k} \quad \forall k \in \mathbb{N}.$$

This implies, $\mu(Y_k) \leq 2^{-k}$ for all $k \in \mathbb{N}$. Define now $Z_j := \bigcup_{k=j}^{\infty} Y_k$ for all $j \in \mathbb{N}$. Then, $\mu(Z_j) \leq 2^{1-j}$ for all $j \in \mathbb{N}$.

Fix $\epsilon > 0$ and choose $j \in \mathbb{N}$ such that $2^{1-j} < \epsilon$. If $x \notin Z_j$ then for $k \geq j$ we have

$$|f_{n_k}(x)| < 2^{-k}.$$

Thus, $\{f_{n_k}\}_{k\in\mathbb{N}}$ converges to 0 uniformly outside of Z_j, where $\mu(Z_j) < \epsilon$. Also, $\{f_{n_k}(x)\}_{k\in\mathbb{N}}$ converges to 0 if $x \notin Z := \bigcap_{j=1}^{\infty} Z_j$. Note that $\mu(Z) = 0$.

In the general case $f \neq 0$ we apply the previous proof to the sequence $\{f_n - f\}_{n\in\mathbb{N}}$. □

Proposition 3.25. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{L}^1(X,\mu)$ converging pointwise to the measurable function f almost everywhere. Then f is integrable and $\{f_n\}_{n\in\mathbb{N}}$ converges to f in the $\|\cdot\|_1$-seminorm.*

Proof. By Theorem 3.17 there exists an integrable function g such that $\{f_n\}_{n\in\mathbb{N}}$ converges to g in the $\|\cdot\|_1$-seminorm. By Theorem 3.24 a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ converges to g pointwise almost everywhere, i.e., outside a set Z_g of measure zero. On the other hand $\{f_n\}_{n\in\mathbb{N}}$ (and any of its subsequences) converges to f almost everywhere, i.e., outside a set Z_f of measure zero. Thus, $f = g$ almost everywhere, i.e., outside the set of measure zero $Z_g \cup Z_f$. By Proposition 3.12, f is integrable. Moreover, $\|f - g\|_1 = 0$ and hence $\{f_n\}_{n\in\mathbb{N}}$ converges to f in the $\|\cdot\|_1$-seminorm. □

Theorem 3.26 (Monotone Convergence Theorem)**.** *Let $\{f_n\}_{n\in\mathbb{N}}$ be a pointwise increasing sequence of real valued functions in $\mathcal{L}^1(X,\mu)$ such that there exists a constant $c \in \mathbb{R}$ with*

$$\int_X f_n \le c \quad \forall n \in \mathbb{N}.$$

Then, the sequence $\{f_n\}_{n\in\mathbb{N}}$ converges to some function $f \in \mathcal{L}^1(X,\mu)$ in the $\|\cdot\|_1$-seminorm and also converges pointwise to f almost everywhere.

Proof. The sequence $\{\int_X f_n\}_{n\in\mathbb{N}}$ is increasing and bounded and thus converges. In particular, it is a Cauchy sequence. But

$$\left|\int_X f_n - \int_X f_m\right| = \int_X |f_n - f_m| = \|f_n - f_m\|_1 \quad \forall n, m \in \mathbb{N},$$

since $\{f_n\}_{n\in\mathbb{N}}$ is pointwise increasing. So, $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the $\|\cdot\|_1$-seminorm. By completeness (Theorem 3.17) there exists a function $f \in \mathcal{L}^1(X,\mu)$ so that $\{f_n\}_{n\in\mathbb{N}}$ converges to f in the $\|\cdot\|_1$-seminorm. By Theorem 3.24 there exists a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ that converges pointwise to f almost everywhere. But, since $\{f_n(x)\}_{n\in\mathbb{N}}$ is increasing for all $x \in X$, it must converge for any $x \in X$ where a subsequence converges. Thus, $\{f_n\}_{n\in\mathbb{N}}$ converges to f almost everywhere. □

Proposition 3.27. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of real valued integrable functions such that there exists a real valued integrable function g with $f_n \le g$ for all $n \in \mathbb{N}$. Then, $\sup_{n\in\mathbb{N}} f_n$ is integrable and,*

$$\sup_{n\in\mathbb{N}} \int_X f_n \le \int_X \sup_{n\in\mathbb{N}} f_n.$$

Proof. Since $\{f_n\}_{n\in\mathbb{N}}$ is bounded pointwise by g, the function $\sup_{n\in\mathbb{N}} f_n$ is well defined. Set $g_n := \sup\{f_1, \ldots, f_n\}$ for all $n \in \mathbb{N}$. Then, $\{g_n\}_{n\in\mathbb{N}}$ is a pointwise increasing sequence of integrable functions. In particular, the g_n

are measurable and so is by Theorem 2.19 their limit $\lim_{n\to\infty} g_n = \sup_{n\in\mathbb{N}} f_n$. Moreover, $\int_X g_n \le \int_X g$ for all $n \in \mathbb{N}$. Thus, we can apply Theorem 3.26 and there exists an integrable function f to which $\{g_n\}_{n\in\mathbb{N}}$ converges pointwise almost everywhere. Thus, $f = \sup_{n\in\mathbb{N}} f_n$ almost everywhere and $\sup_{n\in\mathbb{N}} f_n$ is integrable by Proposition 3.12. For the inequality observe that $f_k \le \sup_{n\in\mathbb{N}} f_n$ for all $k \in \mathbb{N}$. Hence, $\int_X f_k \le \int_X \sup_{n\in\mathbb{N}} f_n$ for all $k \in \mathbb{N}$. Taking the supremum over $k \in \mathbb{N}$ implies the claimed inequality. □

Proposition 3.28 (Fatou's Lemma). *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of real valued integrable functions such that there exists a real valued integrable function g with $f_n \ge g$ for all $n \in \mathbb{N}$. Assume furthermore that $\liminf_{n\to\infty} \int_X f_n$ exists. Then, $f(x) := \liminf_{n\to\infty} f_n(x)$ exists almost everywhere and can be extended to an integrable function on X. Furthermore,*

$$\int_X f \le \liminf_{n\to\infty} \int_X f_n.$$

Proof. Fix $k \in \mathbb{N}$ and apply Proposition 3.27 to the sequence $\{-f_{k+n-1}\}_{n\in\mathbb{N}}$. Thus, $h_k := \inf_{n\ge k} f_n$ is integrable and

$$\int_X h_k \le \inf_{n\ge k} \int_X f_n \le \liminf_{n\to\infty} \int_X f_n \quad \forall k \in \mathbb{N}.$$

But the sequence $\{h_k\}_{k\in\mathbb{N}}$ is increasing and has bounded integral, so we can apply Theorem 3.26. Thus $\{h_k\}_{k\in\mathbb{N}}$ converges pointwise almost everywhere to an integrable function f and

$$\lim_{k\to\infty} \int_X h_k = \int_X f.$$

Thus,

$$\int_X f \le \liminf_{n\to\infty} \int_X f_n.$$

But $f(x) = \lim_{k\to\infty} h_k(x) = \liminf_{n\to\infty} f_n(x)$ almost everywhere. This completes the proof. □

Theorem 3.29 (Dominated Convergence Theorem). *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of integrable functions such that there exists a real valued integrable function g with $|f_n| \le g$ for all $n \in \mathbb{N}$. Assume also that $\{f_n\}_{n\in\mathbb{N}}$ converges pointwise almost everywhere to a measurable function f. Then, f is integrable and $\{f_n\}_{n\in\mathbb{N}}$ converges to f in the $\|\cdot\|_1$-seminorm.*

Proof. Fix $k \in \mathbb{N}$. Consider the set of real valued integrable functions $\{|f_n - f_m|\}_{(n,m)\in I\times I}$ where $I = \{k, k+1, \ldots\}$. Since $|f_n - f_m| \le 2g$ for all $n, m \in I$ we can apply Proposition 3.27 and conclude that $g_k := \sup_{n,m\ge k} |f_n - f_m|$ is integrable. The $\{g_k\}_{k\in\mathbb{N}}$ form a pointwise decreasing sequence and $\int_x g_k \ge 0$. So we can apply Theorem 3.26 to $\{-g_k\}_{k\in\mathbb{N}}$. Since we already know that

$\{g_k\}_{k\in\mathbb{N}}$ converges pointwise to zero almost everywhere we conclude that it also converges to zero in the $\|\cdot\|_1$-seminorm. This implies that $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. (**Exercise.**Show this!) By Proposition 3.25, f is integrable and $\{f_n\}_{n\in\mathbb{N}}$ converges to f in the $\|\cdot\|_1$-seminorm. □

Proposition 3.30. *Let f be a measurable function. Then, f is integrable iff $|f|$ is integrable. Moreover, if $|f| \le g$ for some real valued integrable function g, then f is integrable.*

Proof. By Lemma 3.14 integrability of $|f|$ follows from integrability of f. It remains to show that given g integrable and real valued such that $|f| \le g$, f is integrable. Firstly, since g is integrable, it vanishes outside a σ-finite set A by Proposition 3.13. The same is thus true of f. Let $\{A_n\}_{n\in\mathbb{N}}$ be an increasing sequence of sets of finite measure such that $A = \bigcup_{n\in\mathbb{N}} A_n$. By Theorem 2.23, there is a sequence $\{f_n\}_{n\in\mathbb{N}}$ of simple maps that converges pointwise to f. Define a sequence of maps $\{h_n\}_{n\in\mathbb{N}}$ as follows:

$$h_n(x) := \begin{cases} f_n(x) & \text{if } x \in A_n \text{ and } |f_n(x)| \le 2g(x) \\ 0 & \text{otherwise} \end{cases}$$

It is easy to see that h_n is an integrable simple map for each $n \in \mathbb{N}$. (**Exercise.**Show this!) Moreover, the sequence $\{h_n\}_{n\in\mathbb{N}}$ converges pointwise to f and we have $|h_n| \le 2g$ for all $n \in \mathbb{N}$. Applying Theorem 3.29 shows that f is integrable. □

Proposition 3.31. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of integrable functions converging pointwise almost everywhere to a measurable function f. Assume also that there is a constant $c \in \mathbb{R}$ such that $\|f_n\|_1 \le c$ for all $n \in \mathbb{N}$. Then, f is integrable.*

Proof. $\{|f_n|\}_{n\in\mathbb{N}}$ is a sequence of non-negative valued integrable functions converging pointwise to the measurable function $|f|$. The sequence $\{\int_X |f_n|\}_{n\in\mathbb{N}}$ takes values in the compact interval $[0,c]$ and thus must have a point of accumulation (Proposition 1.31). Together with boundedness from below this implies the existence of $\liminf_{n\to\infty} \int_x |f_n|$ and we can apply Proposition 3.28. By assumption $|f(x)| = \lim_{n\to\infty} |f_n(x)| = \liminf_{n\to\infty} |f_n(x)|$ almost everywhere, so $|f|$ is integrable. By Proposition 3.30, f is integrable. □

3.5 Exercises

Exercise 23 (Lang)**.** Consider the interval $[0,1]$ with the Lebesgue measure μ. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of continuous functions $f_n : [0,1] \to [0,1]$ which converges pointwise to 0 everywhere. Show that

$$\lim_{n\to\infty} \int_0^1 f_n \,\mathrm{d}\mu = 0.$$

Exercise 24 (Lang)**.** Let X, Y be measurable spaces and $f : X \to Y$ a measurable map. Denote the σ-algebra on X by $\mathcal{M}$ and the σ-algebra on Y by $\mathcal{N}$. Let μ be a positive measure on $\mathcal{M}$. Define a function $\nu : \mathcal{N} \to [0, \infty]$ as follows: $\nu(N) := \mu(f^{-1}(N))$. Show that ν is a positive measure on $\mathcal{N}$. Moreover show that if $g \in \mathcal{L}^1(Y, \nu)$, then $g \circ f \in \mathcal{L}^1(X, \mu)$ and

$$\int_X g \circ f \, \mathrm{d}\mu = \int_Y g \, \mathrm{d}\nu.$$

Exercise 25 (Lang, extended)**.** Let X be a measure space with finite measure μ and $f \in \mathcal{L}^1(X, \mu)$. Show that the limit

$$\lim_{n\to\infty} \int_X |f|^{1/n} \, \mathrm{d}\mu$$

exists and compute it. Give an example where the limit does not exist if $\mu(X) = \infty$.

Exercise 26 (Fundamental Theorem of Differentiation and Integration)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and $a, b \in \mathbb{R}$ with $a \le b$. Then,

$$\int_a^b f' \, \mathrm{d}\mu = f(b) - f(a),$$

where μ is the Lebesgue measure. [Hint: Note that f' is integrable on $[a, b]$. Consider the map $g : \mathbb{R} \to \mathbb{R}$ given by $g(y) := \int_a^y f' \, \mathrm{d}\mu$. Show that g is continuously differentiable and that $g' = f'$. Apply the fact that a function with vanishing derivative is constant to the difference $f - g$ to conclude the proof.]

Exercise 27 (Partial Integration)**.** Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and $a, b \in \mathbb{R}$ with $a \le b$. Show that,

$$\int_a^b f g' \, \mathrm{d}\mu = f g|_a^b - \int_a^b f' g \, \mathrm{d}\mu,$$

where $\mathrm{d}\mu$ is the Lebesgue measure.

Exercise 28 (adapted from Lang)**.** Equip the space $[0, \infty]$ with the following topology: A set in $U \subseteq [0, \infty]$ is open iff either U is an open subset of $[0, \infty)$ or $U = \neg A$, where A is a compact subset of $[0, \infty)$.

- Show that this indeed defines a topology on $[0, \infty]$. Moreover, show that this topological space is compact.

- Let X be a measurable space and $f : X \to [0, \infty]$. Let $Y := f^{-1}([0, \infty))$. Show that f is a measurable function iff Y is a measurable set and $f|_Y : Y \to [0, \infty)$ is a measurable function.

- Let X be a measure space with σ-finite measure μ. Show that $f : X \to [0, \infty]$ is measurable iff there exists an increasing sequence $\{f_n\}_{n\in\mathbb{N}}$ of integrable simple functions $f_n : X \to [0, \infty)$ which converges pointwise to f. (Recall that an increasing sequence of real numbers which is not bounded from above is said to converge to ∞.)

- (X and μ as above.) Let $f : X \to [0, \infty]$ measurable. Let $\{f_n\}_{n\in\mathbb{N}}$ be an increasing sequence of integrable simple maps converging pointwise to f. Define the integral of f to be,

  $$\lim_{n\to\infty} \int_X f_n \, \mathrm{d}\mu.$$

  Show that this does not depend on the choice of sequence. Also show that this coincides with the usual definition of integral if $f(X) \subseteq [0, \infty)$ and if f is integrable. Formulate and prove an adapted version of the Monotone Convergence Theorem (Theorem 3.26).

- (X and μ as above.) Let $f : X \to [0, \infty]$ measurable. For each measurable subset $A \subseteq X$ define

  $$\mu_f(A) := \int_A f \, \mathrm{d}\mu.$$

  Show that μ_f is a positive measure. Let $g : X \to [0, \infty]$ measurable and show that,

  $$\int_X g \, \mathrm{d}\mu_f = \int_X fg \, \mathrm{d}\mu.$$